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Nonlinear partial differential equations

Integration of the Harry Dym and Korteweg–de Vries Equations in Parametric Form

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Abstract

In the present work we for the first time apply a relatively new method of constructive unfixed change of variables to the Harry Dym (HD) and the Korteweg–de Vries (KdV) equations. We construct two dynamical systems and formulate necessary conditions for the stability of phase trajectories. A system of functional algebraic equations is constructed and it is proved that two formal solvability conditions for a system of first order partial differential equations have one non–trivial common factor. An important feature of the HD and KdV equations was found: after an unfixed constructive change of variables, a new "hidden" key equation for the function of the partial first derivative can be separated from the other equations. The exact solutions constructed with the help of a non-autonomous dynamical system coincide with global solutions. That is not the case for equations with dissipation. Two classes of exact solutions are found for the HD and for the KdV equation. A possibility arises to construct new asymptotic solutions.

Keywords: the Harry Dym and Korteweg–de Vries equations, dynamical systems, stability of phase paths

1. Introduction

In the 80s of the XX century, in a cycle of works made under the guidance of the member of the USSR Academy of Sciences V.P. Maslov [1–7, 24–34], new original methods were proposed for the analysis of nonlinear partial differential equations (NLPDE's). These works dealt with semilinear parabolic equations, as well as with quasilinear parabolic equations with dissipation and with quadric and cubic nonlinearities. These equations could also contain a small parameter $0 < \varepsilon < 1$.

The solutions studied belong not only to the type of deformed simple waves [1], but also to those arising from their small disturbances. Some solutions to the equations connected with the liquid and gas dynamics, to the Burgers equation in particular, evolve to those close to shock waves. In the limit, as a small parameter tends to 0, they transform into shock waves [3, 6, 7].

An analysis of such solutions for the Burgers equation is presented in detail in Appendix of [1]. Solutions of smoothed shock wave type for semilinear and quasilinear parabolic partial differential equations (PDE) are constructed, in particular, in Appendices of [10–16, 24, 25]. The solutions of the said type have the following interesting feature: they tend to 0 as $t \to -\infty$ on any finite interval of x–axis. According to a figure of speech by V.P. Maslov,"solutions originate from nothing". Such "0 – initial conditions" were used in the inverse problem of scattering, in particular, for Korteweg – de Vries, Kadomtsev–Petviashvili, sine–Gordon NLPDE's [24–34]. In other situation, they were also considered, for example, in [4].

The present study is based on a generalisation of a transformation proposed for the first time in [5]. Consider a quasilinear (actually strongly nonlinear) NLPDE

$$Z'_t - (K(Z)Z'_x)'_x + F(Z) = 0$$

and put

$$K(Z) = \rho(\chi)\ \chi^k > 0,\ Z(x,t) = \chi(\tau),$$

$\tau = x + bt$. Thus we get a first ordinary differential equation (ODE$_1$): $b\,\frac{d\chi}{d\tau} - \frac{d}{d\tau}\left(\rho(\chi)\frac{d\chi^k}{d\tau}\right) - F(\chi(\tau)) = 0$, where $\rho(\chi) > 0,\ \rho(0) > 0,\ \rho(1) > 0,\ k > 1$.

Consider a semilinear (actually also nonlinear) NLPDE

$$u'_t - u''_{xx} + R(u) = 0$$

[1]It is known that equations invariant under the translation group have a solution, commonly called the simple wave, which is described by a function of the argument $Z(x,t) = Z(\tau)\Big|_{\tau=x+bt}$ with the invariant $\tau,\ b = \text{const}$.

and put

$$u(x,t) = \Theta(\xi), \quad \xi = x + bt.$$

Thus we get a second ordinary differential equation (ODE_2): $b\frac{d\Theta}{d\xi} - \frac{d^2\Theta}{d\xi^2} - R(\Theta) = 0$, where $\dfrac{dR}{d\Theta}(\Theta)$ is the continuously differentiable function $\Theta \in [0,1]$, with the conditions $\Theta\Big|_{\xi\to-\infty} \to 0$, $\Theta\Big|_{\xi\to\infty} \to 1$. Then the transformation $\rho(\chi)\frac{d\chi^k}{d\tau} = \frac{d\Theta}{d\xi}(\tau(\chi))$ relates ODE_2 with ODE_1, where the functions of the source and of dissipation $F(\chi), R(\chi)$ are related to each other in the following way: $F(\chi) = \frac{R(\chi)\chi^{1-k}}{k\rho(\chi)}$.

This transformation was used in the cited works to study the reference equation with the scope to establish properties of asymptotic solutions [5–7, 24, 25]. The idea of the method called by us “the method of unfixed constructive change of variables” (the UCCV method) was for the first time proposed in [10]. In the cited above second order quasilinear parabolic partial differential equation QLPDE, the variables of the unknown function $Z(x,t)$ and its first partial derivatives, as well as the independent variables $x = x(\xi,\delta)$, $t = t(\xi,\delta)$ were changed by arbitrary twice continuously differentiable functions [10–16].

At first, three relations were studied. Namely, two arbitrary functions were introduced to change the partial derivatives $\frac{\partial Z(x,t)}{\partial t}$, $\frac{\partial Z(x,t)}{\partial x}$ in new independent variables. One more relation follows from the QLPDE. They generalize the described above transformation for the case of two independent variables. These relations were studied in [10, 7]. Later, these relations were completed by the fourth condition of the equality of mixed derivatives of the unknown function in both old and new variables for different QLPDE's for the case of two independent variables. These relations were studied in [11–16].

Note that the addition of the fourth relation cardinally changes the situation. In [11–16], it is proved that the four relations make a system of functional linear algebraic equations (SFLAE's) in new variables. The derivatives of the old variables $x = x(\xi,\delta)$, $t = t(\xi,\delta)$ with respect to the new ones, ξ, δ, are chosen as new variables.

The work [15] also gives a solution of a model optimal control problem for the Hamilton–Jacobi–Bellman equation. The solution is constructed by the UCCV method for the case of three independent variables. In a simplified way, it is described in [22, 23].

Problems connected with the KdV equation have long been subject of interest of the scientific school by the academician V.P. Maslov. Asymptotic solutions were studied in [3, 32–34] and in a number of V.P.Maslov's works made together with his pupils [26–34]. For example, in [6, p. 180], [7, pp. 47–51], a rational soliton solution and a two-soliton solution of the KdV equation were found. The UCCV method was defended as a D.Sc. thesis [15]. Formulas for a nonlinear dynamical system (NLDS) and variants of SFLAE's for the KdV equation were for the first time obtained in [18–20]. In the cycle of works [8–9, 17], other authors developed methods applicable to different NLPDE's, such as the HD, the KdV equations, and others.

Our work is based on a new method (the UCCV method), which, when applied to the HD and KdV equations, permits us to reveal, separately for each of them, a "hidden" new key NLPDE that allows us to construct new solutions. The main results of this work were presented in [21].

2. A system of functional linear algebraic equations for the KdV equation

Consider together the whole set of the KdV equations:

$$\frac{\partial Z(x,t)}{\partial t} + Z^n\,\frac{\partial Z(x,t)}{\partial x} + \beta\,\frac{\partial^3 Z}{\partial x^3} = 0. \tag{1}$$

Commonly, one distinguishes the case of $n = 1$ (weak dispersion) and that of $(n = 2)$ corresponding to a modified KdV equation and to a strong dispersion. Let's assume that the all functions involved are smooth, namely, they are trice continuously differentiable in their arguments.

It is well known [8–9] that the group of translation transformations $Z(x,t) = u(\theta)$ with the invariant $\theta = x - Vt$ allows one to reduce the KdV equation to a third order ODE. The first integral of this ODE describes a nonlinear anharmonic oscillator. Multiplying the second order ODE obtained by the first derivative $u'(\theta)$ and integrating once more, we get

$$(u'(\theta))^2 = E - 2\,(u(\theta))^{n+2}/(\beta(n+1)(n+2)) + V\,(u(\theta))^2/\beta - C_1\,u(\theta),$$

$\beta,\ n \neq -1, -2$. The last equation means that an ODE for the function of the first derivative may be separated from the other equations. In the case of an ODE, this conclusion is trivial and well known. It is known that, at different values of the constants, the cited ODE has the C. Jacobi soliton solution or different solutions describing nonlinear oscillations.

In the present work we found variables for the HD and KdV equations, in which an equation for the first derivative can be separated from the other equations. We call this NLPDE a "hidden" one. All other functions can be expressed in terms of the solution of this equation. This new property of the said equations is connected with integrability. Then the initial NLPDE is integrable in parametric form. It is important that these new in the theory of the HD and KdV equations nonlinear differential equations for the functions of the first derivative are partial differential equations. This permits us to construct exact and asymptotic formulas for the first derivative, the function $Y(\xi,\delta)$. All other functions (derivatives) can be expressed in $Y(\xi,\delta)$.

Let's apply the UCCV method to equation (1). Let's make an unfixed constructive change of variables for the solution of this equation $Z(x,t)$ using a smooth unknown function of new variables $U(\xi,\delta)$:

$$Z(x,t)\Big|_{x=x\ (\xi,\ \delta),\ t=t\ (\xi,\ \delta)} = U(\xi,\delta). \tag{2}$$

Suppose that the Jacobian of the transformation $\det J = x'_\xi t'_\delta - t'_\xi x'_\delta$ is neither zero nor infinity (otherwise this method is not applicable). Then there exists, at least locally, an inverse transformation $\xi = \xi(x,t), \delta = \delta(x,t)$, and $Z(x,t) = U(\xi,\delta)\Big|_{\xi=\xi(x,t),\ \delta=\delta(x,t)}$.

The Jacobian matrix of the inverse transformation has the form

$$J^{-1} = \begin{pmatrix} \xi'_x(x,t) & \delta'_x(x,t) \\ \xi'_t(x,t) & \delta'_t(x,t) \end{pmatrix}.$$

The equality $JJ^{-1} = E$ must be satisfied. The derivatives of old variables x,t with respect to new variables ξ,δ are connected by the relations

$$\frac{\partial x}{\partial \xi} = \det J\frac{\partial \delta}{\partial t},\ \frac{\partial t}{\partial \xi} = -\det J\frac{\partial \delta}{\partial x},\ \frac{\partial x}{\partial \delta} = -\det J\ \frac{\partial \xi}{\partial t},\ \frac{\partial t}{\partial \delta} = \det J\ \frac{\partial \xi}{\partial x}. \tag{3}$$

Analogously, we make three more changes of variables:

$$\begin{aligned}
&\frac{\partial Z}{\partial t}\Big|_{x=x(\xi,\ \delta),\ t=t\ (\xi,\ \delta)} = T(\xi,\delta)\\
&\frac{\partial Z}{\partial x}\Big|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = Y(\xi,\delta),\\
&\frac{\partial Y(\xi(x,t),\delta(x,t))}{\partial x}\Big|_{x=x(\xi,\ \delta),\ t=t(\xi,\ \delta)} = M(\xi,\delta).
\end{aligned} \tag{4}$$

Using (2) and (3),(4), we get the three equations

$$-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \delta} + \frac{\partial U}{\partial \delta}\frac{\partial x}{\partial \xi} = T(\xi,\delta)\ (x'_\xi\ t'_\delta - t'_\xi\ x'_\delta), \tag{5}$$

$$\frac{\partial U}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial U}{\partial \delta}\frac{\partial t}{\partial \xi} = Y(\xi,\delta)\ (x'_\xi\ t'_\delta - t'_\xi\ x'_\delta), \tag{6}$$

$$\frac{\partial Y}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial Y}{\partial \delta}\frac{\partial t}{\partial \xi} = M(\xi,\delta)\ (x'_\xi\ t'_\delta - t'_\xi\ x'_\delta). \tag{7}$$

Equation (1) in new variables takes the form

$$\begin{aligned}&\frac{\partial M}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial M}{\partial \delta}\frac{\partial t}{\partial \xi} = -T_1(\xi,\delta)\ (x'_\xi\ t'_\delta - t'_\xi\ x'_\delta)/\beta,\\ &T_1(\xi,\delta) = T(\xi,\delta) + U(\xi,\delta)^n\ Y(\xi,\delta),\end{aligned} \tag{8}$$

Necessarily, for the function $Z(x,t)$, the equality of mixed derivatives

$$Z''_{tx} = Z''_{xt} \tag{9}$$

must hold in new variables $\xi,\ \delta$.

Using (2)–(4), we get one more equation

$$\frac{\partial x}{\partial \delta}\frac{\partial Y}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial Y}{\partial \delta} + \frac{\partial t}{\partial \delta}\frac{\partial T}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial T}{\partial \delta} = 0. \tag{10}$$

The KdV equation is equivalent to the system of NLPDE's (5)–(8), (10).

We will analyse this system in two stages. At the first stage, we consider the system of NLPDE's (5),(6),(8),(10) as an algebraic system with respect to the derivatives

$$x'_\xi,\ x'_\delta,\ \ t'_\xi,\ t'_\delta. \tag{11}$$

Theorem 1.

Let the system of the four NLPDE's (5), (6), (8), (10) with respect to the variables (11) be given. Then it is a system of linear functional algebraic equations with respect to the variables (11) and it has a unique nontrivial solution

$$\begin{aligned}\frac{\partial x}{\partial \xi} &\stackrel{\text{def}}{=} z_1(\xi,\delta) = (\beta\ Y(\xi,\delta)\ \big(T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\big)\, U'_\xi(\xi,\delta) +\\ &+T_1(\xi,\delta)\ \big(T'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\big)\, U'_\xi(\xi,\delta) +\\ &+\beta\ T(\xi,\delta)\ \big(M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\big)\, Y'_\xi(\xi,\delta))/\Psi,\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial x}{\partial \delta} &\stackrel{\text{def}}{=} z_2(\xi,\delta) = (\beta\ Y(\xi,\delta)\ \big(T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\big)\ U'_\delta(\xi,\delta) +\\ &+T_1(\xi,\delta)\ \big(T'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\big)\ U'_\delta(\xi,\delta) +\\ &+\beta\ T(\xi,\delta)\ \big(M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\big)\ Y'_\delta(\xi,\delta))/\Psi,\end{aligned} \tag{13}$$

$$\frac{\partial t}{\partial \xi} \stackrel{\text{def}}{=} z_3(\xi,\delta) = \left(\beta\ Y(\xi,\delta)\ M'_\xi(\xi,\delta) + T_1(\xi,\delta)\ U'_\xi(\xi,\delta)\right) \times$$
$$\times \left(Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)/\Psi, \tag{14}$$

$$\frac{\partial t}{\partial \delta} \stackrel{\text{def}}{=} z_4(\xi,\delta) = (\beta\ Y(\xi,\delta)\ M'_\delta(\xi,\delta) + T_1(\xi,\delta)\ U'_\delta(\xi,\delta)) \times$$
$$\times \left(Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta) - Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta)\right)/\Psi,$$
$$\Psi = \beta\ Y^2 \left(T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta) - T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta)\right) +$$
$$+T\ T_1 \left(Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta) - Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta)\right) +$$
$$+Y\ (T_1\ \left(U'_\xi(\xi,\delta)\ T'_\delta(\xi,\delta) - U'_\delta(\xi,\delta)\ T'_\xi(\xi,\delta)\right) +$$
$$+\beta\ T\ \left(Y'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta) - Y'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta)\right)). \tag{15}$$

The Jacobian has the form

$$\det J = (\beta\ \Big(M'{}_\xi(\xi,\delta)\ U'_\delta(\xi,\delta) - M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta)\Big) \times$$
$$\times \Big(U'{}_\delta(\xi,\delta)\ Y'_\xi(\xi,\delta) - U'_\xi(\xi,\delta)\ Y'_\delta(\xi,\delta)\Big))/\Psi. \tag{16}$$

The function $T(\xi,\delta)$ is found from (7) and has the form

$$T(\xi,\delta) = (\beta\ M(\xi,\delta)\ \left(M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta) - \beta\ M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta)\right) +$$
$$+Y(\xi,\delta)\ (-Y'_\delta(\xi,\delta)\ \left(U'_\xi(\xi,\delta)\ U^n + \beta\ M'_\xi(\xi,\delta)\right) +$$
$$+Y'_\xi(\xi,\delta)\ (\ U'_\delta(\xi,\delta)\ U^n + \beta\ M'_\delta(\xi,\delta)))))/\Psi_0,$$
$$\Psi_0 = Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta). \tag{17}$$

Proof. At first, using elementary transformations, as in the Gaussian method for solving systems of linear algebraic equations, we eliminate the Jacobian from equations (5), (6), (8) and get two linear equations. Equation (10) is linear. Using (11), we express any three derivatives of old variables with respect to new ones and substitute them into the remaining fourth equation. After the reduction of similar terms, the resulting equation turns out to be linear. Then we get (12)–(15). Afterwards, we calculate (16), (17). Substituting the function of the Jacobian into the right sides of equations (5)–(8), we get a classical form of the SFLAE.

Theorem 1 is proved.

3. The possibilities to study the stability of solutions

Let the system of equations (12)–(15) be given. One can construct the first nonlinear dynamical system (NLDS) (12), (14), and the second NLDS

(13), (15). Actually, new variables ξ and δ may be interpreted as a new time along trajectories.

a. The system (12), (14) may be considered as a NLDS. It has a fixed point (14), in which there is no degeneration (16): $\det J \neq 0$.

$$M'_\xi(\xi,\delta) = -U'_\xi(\xi,\delta)\; T_1/(\beta\; Y). \tag{18}$$

Thus $t'_\xi(\xi,\delta) = 0$. Then $t(\xi,\delta) = \tau(\delta)$. From (6) we obtain $x'_\xi(\xi,\delta) = U'_\xi(\xi,\delta)/Y(\xi,\delta)$ and from (7) it follows that

$$M(\xi,\delta) = Y'_\xi\; Y(\xi,\delta)/U'_\xi(\xi,\delta). \tag{19}$$

Equating (18) to the partial derivative of (19) with respect to the variable ξ, we get

$$T(\xi,\delta) = -U^n\; Y - \beta\; Y\; \left(\frac{\partial}{\partial\;\xi}\left(Y\; Y'_\xi/U'_\xi\right)\right)/U'_\xi. \tag{20}$$

Then we can calculate the elements of a new Jacobian matrix J_1 for the system (12), (14), its eigenvalues and trace, a new Jacobian, as well as the information on the phase flow and other characteristics.

b. The system (13), (15) may be also considered to be a NLDS. It also has a fixed point (15), in which there is no degeneration: $\det J \neq 0$.

Analogously to case (**a**) we have

$$M'_\delta(\xi,\delta) = -U'_\delta(\xi,\delta)\; T_1/(\beta\; Y). \tag{21}$$

Then $t'_\delta(\xi,\delta) = 0$. Let $t(\xi,\delta) = \xi$. From (6) we obtain $x'_\delta(\xi,\delta) = U'_\delta(\xi,\delta)/Y(\xi,\delta)$, and from (7) it follows that

$$M(\xi,\delta) = Y'_\delta\; Y(\xi,\delta)/U'_\delta(\xi,\delta). \tag{22}$$

Equating (21) to the partial derivative of (22) with respect to δ, we get

$$T(\xi,\delta) = -U^n\; Y - \beta\; Y\; \left(\frac{\partial}{\partial\delta}(Y\; Y'_\delta/U'_\delta)\right)/U'_\delta. \tag{23}$$

Taking into consideration the equality $t'_\xi = 1$, we can write equation (14) in the form

$$\frac{\partial}{\partial\delta}(\; T(\xi,\delta)/Y(\xi,\delta)) + \left(Y'_\delta\; U'_\xi - Y'_\xi\; U'_\delta\right)/Y^2 = 0. \tag{24}$$

Then one can calculate the elements of a new Jacobian matrix J_2 for the system (13), (15), etc. Note a symmetry in (18)–(20) and (21)–(23) under the change of ξ by δ. The study of the stability of solutions to the KdV equation is out of the scope of the present work.

4. The property of solvability conditions for the system (12)–(15)

At the second stage, we consider a system of first order NLPDE's with respect to the functions $x = x(\xi, \delta)$, $t = t(\xi, \delta)$. It is well known that the solvability condition of such type of system is the equality of mixed second partial derivatives of the functions $x = x(\xi, \delta)$ and $t = t(\xi, \delta)$ with respect to the arguments ξ and δ. Equations (12)–(15) are a basis to calculate these derivatives. We have then

$$\frac{\partial^2 x(\xi,\delta)}{\partial\xi\,\partial\delta} = \frac{\partial^2 x(\xi,\delta)}{\partial\delta\,\partial\xi},\ \frac{\partial^2 t(\xi,\delta)}{\partial\xi\,\partial\delta} = \frac{\partial^2 t(\xi,\delta)}{\partial\delta\,\partial\xi}. \tag{25}$$

The following theorem is one of the central results of the UCCV method.
Theorem 2. Let the system of equations (12)–(15) be given. Then

a. we have two solvability conditions

$$\frac{\partial z_1(\xi,\delta)}{\partial\delta} - \frac{\partial z_2(\xi,\delta)}{\partial\xi} = TQ = 0,\ \frac{\partial z_3(\xi,\delta)}{\partial\delta} - \frac{\partial z_4(\xi,\delta)}{\partial\xi} = Y{\Psi_0}^2\, Q = 0, \tag{26}$$

for which there exists a nontrivial common multiplyer $Q(\xi, \delta)$ for any smooth function $M(\xi, \delta)$.

b. the two solvability conditions (12)–(15) reduce to an equality to zero of the common multiplier

$$Q(\xi,\delta) = 0 \tag{27}$$

for any smooth functions $U(\xi, \delta)$, $Y(\xi, \delta)$, $T(\xi, \delta)$, $M(\xi, \delta)$.

Proof. This feature of NLPDE's was not known before. It was published for the first time in [11] for the NLPDE cited in the Introduction. A new NLPDE $Q = 0$ depends on the unknown functions $Y(\xi, \delta)$, $T(\xi, \delta)$, $M(\xi, \delta)$, $U(\xi, \delta)$ and their first derivatives. Equation (27) can be obtained by direct calculations of (26). In its initial form, this equation is very cumbersome [2], however, after the construction of an exact solution, it becomes essentially simpler and, as a result, we get a “hidden” key equation containing a sum of only three terms. The equation of the type indicated in the formulation of the theorem for the

[2] It occupies more that three pages in font 14. To check it we used the mathematical symbolic computation program Wolfram Mathematica.

KdV equation in the first case was for the first time proposed in a report at a conference [21] and is published for the first time in the present work. If any four smooth functions U, Y, T, M satisfy (27), then the system (12)–(15) and (5), (6), (8), (10) is solvable. With the added equation (7) the solution of the initial NPDE KdV (1) is recovered by (2). *Theorem 2 is proved.*

5. Exact solutions of the Korteweg–de Vries equation.

The construction, for the functions $U(\xi,\delta),\ Y(\xi,\delta), T(\xi,\delta),\ M(\xi,\delta)$, of exact solutions to the KdV equation by the UCCV method allows us to prove that if the function of the first derivative $Y(\xi,\delta)$ is found, then all the other functions can also be found. Then we can return to the initial solutions (2)–(4) of the KdV equation (1) in parametric form.

Theorem 3.

Let the system (5)–(8), (10) be given. Then the change of variables (2)–(4) allows us to obtain, in new variables, an equation for the function of the first derivative $Y(\xi,\delta)$. This equation may be separated from all the other equations and depends only on the function $U(\xi,\delta)$ and its derivatives:

$$\frac{\partial}{\partial\delta}\left(U^n(\xi,\delta)+\beta\left(\frac{\partial}{\partial\delta}\ (Y\ Y'_\delta/U'_\delta\ )\right)/U'_\delta\ \right)=(Y'_\delta\ U'_\xi-Y'_\xi\ U'_\delta)/Y^2. \tag{28}$$

One more change of variables for the function $Y(\xi,\delta)$ has the form [3]

$$Y(\xi,\delta)=\sqrt{G(\eta,\xi)}\Big|_{\eta=U(\xi,\delta)},\ \eta=U(\xi,\delta),\ t(\xi,\delta)=\xi. \tag{29}$$

Thus, a new class of exact solutions of the KdV equation is described by the formulas

$$\begin{aligned}
&M(\xi,\delta)=G'_\eta(\eta,\xi)/2\Big|_{\eta=U(\xi,\delta)},\\
&T(\xi,\delta)=-\sqrt{G(\eta,\xi)}\ (2\ \eta^n+\beta\ G''_{\eta\eta}(\eta,\xi))/2\Big|_{\eta=U(\xi,\delta)},\\
&x(\xi,\delta)=X(\eta,\xi)\Big|_{\eta=U(\xi,\delta)},\\
&\frac{\partial\ X(\eta,\xi)}{\partial\tau}=\eta^n+\beta\ G''_{\eta\eta}/2,\quad \frac{\partial\ X(\eta,\xi)}{\partial\ \eta}=1/\sqrt{G(\eta,\xi)},\\
&\det J=-U'_\delta(\xi,\delta)/G(\eta,\xi)^{1/2}\Big|_{\eta=U(\xi,\delta)},
\end{aligned} \tag{30}$$

[3]In the present work, for the sake of simplicity in the presentation of formulas, they are given for a positive branch of the solution,except for an example, in which they need to be presented otherwise.

and the function $G(\eta, \xi)$ can be found from the equation

$$\frac{\partial G(\eta,\xi)}{\partial \xi}/(\beta\ G(\eta,\xi)^{3/2}) + \frac{\partial^3\ G(\eta,\xi)}{\partial \eta^3} + 2n\eta^{n-1}/\beta = 0. \tag{31}$$

new in the theory of the KdV equation.

The Jacobian has the form $\det J = -U'_\delta/\sqrt{G(\eta,\tau)}\Big|_{\eta=U(\xi,\delta)}$. The function $U(\xi,\delta)$ remains an arbitrary, smooth, and trice continuously differentiable function.

Proof. Consider case (**b**) in Section 3. Let $t(\xi,\delta) = \xi$, then we obtain relations (21)–(24). Then the solvability condition (25) reduces to equation (24). The equation $t'_\xi = 1$ follows from (14) and also takes the form (24). Combining relations (23) and (24)and excluding the function T, we get (24). After the change of variables (29) we again obtain equation (31). From (12), (13) we get the following relations for derivatives: $x'_\xi(\xi,\delta) = \Big(U'_\xi/\sqrt{G(\eta,\xi)} + \eta^n + \beta\ G''_{\eta\eta}/2\Big)\Big|_{\eta=U(\xi,\delta)}$, $x'_\delta(\xi,\delta) = U'_\delta/\sqrt{G(\eta,\xi)}\Big|_{\eta=U(\xi,\delta)}$.

Put $x(\xi,\delta) = X(U(\xi,\delta),\xi)$. Then relations (30) hold. For the solvability of (25) in the variables $\eta,\ \xi$, the relations $\frac{\partial^2 X(\eta,\xi)}{\partial\eta\ \partial\xi} = \frac{\partial^2 X(\eta,\xi)}{\partial\xi\ \partial\eta}$ must hold. Then we again obtain equation (31). This means that the exact solution (28)–(31) does exist. *Theorem 3 is proved.*

Example. Now we briefly describe a new non-trivial exact solution of (31) and, consequently, also of (1). At $n = 1$, equation (31) has the solution $G(\eta,\xi) = = \Xi(\theta)$ with the invariant $G(\eta,\xi) = = \Xi(\theta)$. Then (31) implies an ordinary differential equation (ODE_3). The first integral of ODE_3 has the form ODE_4:

$$\Xi''(\theta) \mp 2V/(\beta\sqrt{\Xi(\theta)}) = -C_4 - 2\theta/\beta,$$

Here C_4 is a constant. This is a nonlinear ODE with a right–hand side. One can multiply ODE_4 by the function $\Xi'(\theta)$. After the integration we get the quadrature

$$(\Xi'(\theta))^2/2 + C_5 + C_4\Xi(\theta) + 2(\theta\Xi - \int \Xi(\theta)\, d\theta)/\beta - 4V\sqrt{\Xi(\theta)}/\beta = 0,$$

Here C_5, C_4 are constants. Put $X(\eta,\xi) = X_1(\eta + V\xi)$. Then (30) implies $X(\theta) = \pm\int(1/\sqrt{\Xi(\theta)})\, d\theta$. This is an implicit equation for the calculation of the function $\Xi(\theta)$.

Note, besides, that ODE_4 implies a non-autonomous dynamical system $\Xi'(\theta) = p(\theta),\ p'(\theta) = -C_4 - 2\theta/\beta \pm 2V/\beta\sqrt{\Xi(\theta)})$.

Let's calculate

$\frac{d\theta}{d\Xi(\theta)} = 1/\left(\frac{d\Xi(\theta)}{d\theta}\right) = 1/p(\theta)$. Then the Jacobian matrix$_2$ has the form

$$\begin{pmatrix} 0 & 1 \\ -2/(\beta p) \pm V/(\beta \Xi^{3/2}) & 0 \end{pmatrix}.$$

A fixed point is determined by the expressions $p = 0,\ \Xi = 4V^2/(C_4\beta + 2\theta)^2$. Hence it follows that on the equator of the Poincaré sphere there can exist,at different values of parameters, critical points of center or saddle type. Then one should analyse the existence of bifurcations. A detailed study shows that the obtained new exact solution describes an asymmetrical wave with a big value of the derivative on the leading edge that differs from C.Jacobi's solution. This solution is mentioned in Section 2 and is studied in detail, for instance, in [8, 9, 17].

Obviously, a complete study of this solution should be the subject of a separate work. The other solution has the form $G(\eta, \xi) = \eta^{n+2}(\Psi(\varphi))',\ \ \varphi = \eta^{3ns/2}\xi^s,\ \ s \neq 0$.

Note that equations (28), (31) are new in the theory of the KdV equation.

Remark 1. For the Korteweg–de Vries–Burgers equation [1–3, 9, 17], in which second derivatives are added to equation (1) (dissipation), there is no solution constructed by the UCCV method, which would be analogous to the solution constructed in Theorem 3. When proving Theorem 3 we three times obtain the same equation (28), (31). In the case of the equation with dissipation it is not so, because we obtain equations that differ in the terms connected with dissipation.

6. Possibility to construct, with functional arbitrariness, asymptotic solutions of equation (31) and, on the whole, of the KdV equation.

Remark 2. Let's show that, in a degenerate case, the formulas of Theorem 3 imply classic results. The Jacobian (16) equals zero on the group of translation transformations $U(\xi, \delta) = U_0(\delta - V\xi)$, or if the function of the first derivative $Y(\xi, \delta) =$ $= Y_0(U(\xi, \delta))$ depends only on the function $Y(\xi, \delta) =$ $= Y_0(U(\xi, \delta))$. This is a degenerate case, and the UCCV method is not applicable here.

If $Y(\xi, \delta) = Y_0(U(\xi, \delta))$, then $G(\eta, \xi) = G(\eta)$. Thus from (28), (31) we obtain the equation for the function of the first derivative $Y(\xi, \delta)$ (4)

$Y^2(\eta) = -2\ \eta^{2+n}/(\beta(n+1)(n+2)) + C_1 + C_2\ \eta + C_3\ \eta^2,\ n \neq -1, -2.$

This is an analogue of the ODE in the beginning of section 2.

As an example, we choose a draft for the construction of an approximate, asymptotic in a small parameter $0 < \varepsilon < 1$, solution of equation (31) using a method similar to the Poincaré–Lighthill–Kuo method, which was actively applied in [6, 7] in the form

$$G(\eta, \xi, \varepsilon) = -2\eta^{n+2}/(\beta(n+1)(n+2))\ (1 + s\ \varepsilon\ \eta^{\sigma-n-2}\psi(\eta, \xi) + O(\varepsilon^2)) + \\ +C_1(1 + \varepsilon\ \eta^{\sigma}\rho(\eta, \xi) + O(\varepsilon^2)) + C_2\eta\ (1 + \varepsilon\ \eta^{-1+\sigma}\varphi(\eta, \xi) + O(\varepsilon^2)) + \\ +C_3\eta^2\ (1 + \varepsilon\ \eta^{-2+\sigma}\mu(\eta, \xi) + O(\varepsilon^2)). \qquad (32)$$

Here s, σ are constants. There exist correction functions $\psi(\eta, \xi)$, $\rho(\eta, \xi)$, $\varphi(\eta, \xi)$, $\mu(\eta, \psi)$ of the order ε, which satisfy a linear PDE. The coefficient in these functions can be chosen by different ways.

When solving this linear PDE, one can express one of the functions in terms of the others. The arbitrariness in the choice of such function can be used to find a solution of the Cauchy problem. One can consider a variant, in which one should equate all the four correction functions. A detailed calculation of asymptotic solutions of the KdV equation deserves a separate study.

7. A second class of exact solutions to the Korteweg–de Vries equation We describe this class only briefly. It is more complicated than the first one.

Theorem 4. Let the system (5)–(8), (10) be given. Then an exact solution of the KdV equation (1) has the form

$$Y(\xi, \delta) = \sqrt{G(\eta, \delta)}\Big|_{\eta=U(\xi,\delta)},\ \eta = U(\xi, \delta),\ x(\xi, \delta) = \delta, \qquad (33)$$

$$M(\xi, \delta) = M_0(\eta, \delta)\Big|_{\eta=U(\xi,\delta)},\ t(\xi, \delta) = \tau(\eta, \delta)\Big|_{\eta=U(\xi,\delta)}, \qquad (34)$$

$$M_0(\eta, \delta) = G'_\delta(\eta, \delta)/(2\ \sqrt{G(\eta, \delta)}) + G'_\eta(\eta, \delta)/2. \qquad (35)$$

The function $T(\xi, \delta)$ is given in (17). After the change of variables $T(\xi, \delta)$ we get $T(\xi, \delta) = T_0(U(\xi, \delta), \delta)\Big|_{U(\xi,\delta)=\eta}$. Then $T_0(\eta, \delta) = -\left(\sqrt{G(\eta, \delta)}(\beta M'_{0\ \delta}(2M_0 - G'_\eta(\eta, \delta)) + G'_\delta(\eta, \delta)(\eta^n + \beta M'_{0\ \eta})/G'_\delta(\eta, \delta)\right)$

$= -\eta^n\sqrt{G(\eta,\delta)} + \beta(G'_\delta)^2/(4\ \sqrt{G^3}) - \beta G''_{\delta\delta}/(2\sqrt{G}) + \beta\frac{\partial G(\eta,\delta)}{\partial\delta}\frac{\partial G(\eta,\delta)}{\partial\eta}/(4G)-$
$-\beta\frac{\partial^2 G(\eta,\delta)}{\partial\eta\partial\delta} - \beta\sqrt{G}\frac{\partial^2 G(\eta,\delta)}{\partial\eta^2}/2.$

Then the equation for the function of the first derivative $Y(\xi,\delta) = \sqrt{G(\eta,\delta)}\Big|_{\eta=U(\xi,\delta)}$ $Y(\xi,\delta) = \sqrt{G(\eta,\delta)}\Big|_{\eta=U(\xi,\delta)}$ in new variables can be separated from the other equations. It depends only on the function $U(\xi,\delta)$ and its derivatives:

$$\frac{\partial^3 G(\eta,\delta)}{\partial\eta^3} + 3\frac{\partial^3 G(\eta,\delta)}{\partial\eta\partial\delta^2}/G + 3\frac{\partial^3 G(\eta,\delta)}{\partial\eta^2\partial\delta}/\sqrt{G(\eta,\delta)} -$$
$$-3\left(G'_\delta + \sqrt{G}G'_\eta\right)\frac{\partial^2 G(\eta,\delta)}{\partial\eta\partial\delta}/(2G^2) + 3\frac{\partial G(\eta,\delta)}{\partial\delta}\left(\frac{\partial G(\eta,\delta)}{\partial\eta}\right)^2/(4\sqrt{G^5}) -$$
$$-3\left(-(G'_\delta)^2 + GG''_{\eta\eta}\right)\frac{\partial G(\eta,\delta)}{\partial\eta}/(2G^3) + \frac{\partial^3 G(\eta,\delta)}{\partial\delta^3}/\sqrt{G^3} +$$
$$+2n\eta^{n-1}/\beta - 3G'_\delta G''_{\delta\delta}/(2\sqrt{G^5}) + 3(G'_\delta)^3/(4\sqrt{G^7}) + \eta^n G'_\delta/(\beta\sqrt{G^3}) = 0. \quad (36)$$

The derivatives of the function $\tau(\eta,\delta)$ have the form $\tau'_\delta(\eta,\delta) = 4\ G^2(\eta,\delta)/\Psi_2$, $\quad \tau'_\eta(\eta,\delta) = -4\sqrt{G^3(\eta,\delta)}/\Psi_2$,
$\Psi_2 = 4\eta^n G^2(\eta,\delta) - \beta(G'_\delta)^2 + 2\beta GG''_{\delta\delta} - \beta\sqrt{G}G'_\delta G'_\eta + 4\beta\sqrt{G^3}G''_{\delta\eta} + 2\beta G^2 G''_{\eta\eta}.$

The function $U(\xi,\delta)$ is an arbitrary trice continuously differentiable function.

Proof. Put $x(\xi,\delta) = \delta$, $x'_\xi(\xi,\delta) = 0$, $x'_\delta(\xi,\delta) = 1$. Then (12),(13) imply equations

$$\beta\ Y(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\right)U'_\xi(\xi,\delta) +$$
$$+T_1(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)U'_\xi(\xi,\delta) +$$
$$+\beta\ T(\xi,\delta)\ \left(M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)Y'_\xi(\xi,\delta) = 0, \quad (37)$$

$$(\beta\ Y(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\right)\ U'_\delta(\xi,\delta) +$$
$$+T_1(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)\ U'_\delta(\xi,\delta) +$$
$$+\beta\ T(\xi,\delta)\ \left(M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)\ Y'_\delta(\xi,\delta))/\Psi = 1, (38)$$

The expression for the function $T(\xi,\delta)$ follows from (17). Using (37),(38), after the change of variables (37),(38) we get relation (35). Taking into account

(37)and(38), after the change of variables (33)-(35), using (36), we obtain relations connecting the derivatives $\tau'_\delta(\eta,\delta)$, $\tau'_\eta(\eta,\delta)$. Then we make some simplifications. Let's express the third derivative $\frac{\partial^3 G(\eta,\delta)}{\partial\eta^3}$ by using (36) and eliminating the third derivative from the relation between first derivatives $\tau'_\delta(\eta,\delta)$, $\tau'_\eta(\eta,\delta)$.

In the present case, besides the solvability condition (25) in variables η, δ, the condition $\frac{\partial^2\tau(\eta,\delta)}{\partial\eta\ \partial\delta} = \frac{\partial^2\tau(\eta,\delta)}{\partial\delta\ \partial\eta}$ must hold. Here, after the change of variables (33)– (35) we again get equation (36). This means that the exact solution (33)–(36) does exist.

From the equation $x'_\delta = 1$, which has the form (38), with the help of (33)–(35), we again obtain equation (36). *Theorem 4 is proved.*

Note that, when proving Theorem 4, we trice obtain the same equation (36). This does not occur in the case of the Korteweg–de Vries–Burgers equation with dissipation.

8. A new class of exact solutions to the Harry Dym equation.

In analogy with Teorems 1–4, we for the first time apply the UCCV method to the Harry Dym equation.

$$\frac{\partial Z(x,t)}{\partial t} + Z^\alpha \frac{\partial^3 Z}{\partial x^3} = 0. \tag{39}$$

Let's make the change of variables (2)–(7). Equation (8) follows from (39) and has the form

$$\frac{\partial M}{\partial\xi}\frac{\partial t}{\partial\delta} - \frac{\partial M}{\partial\delta}\frac{\partial t}{\partial\xi} = -T(\xi,\delta)\ \det J\ U^{-\alpha}. \tag{40}$$

Thus we obtain the system (5), (6), (40), (10). Theorems analogous to Theorems 1–4 are true.

Theorem 5.

Let the system of the four equations (5), (6), (40), (10) with respect to the derivatives (11) be given. Then the SFLAE (5), (6), (40), (10) has a unique solution

$$\frac{\partial x}{\partial\xi} = ((Y(\xi,\delta)\ \left(M'_\delta(\xi,\delta)\ T'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ T'_\delta(\xi,\delta)\right) U'_\xi(\xi,\delta) +$$
$$+T(\xi,\delta)\ \left(U'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - U'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\right)\ U^\alpha + T(\xi,\delta)\ U'_\xi(\xi,\delta)\times$$
$$\times\left(U'_\delta(\xi,\delta)\ T'_\xi(\xi,\delta) - U'_\xi(\xi,\delta)\ T'_\delta(\xi,\delta)\right))/\Psi_1, \tag{41}$$

$$\frac{\partial x}{\partial \delta} = (T(\xi,\delta)\ U'_\delta(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right) +$$
$$+U^\alpha\ (Y(\xi,\delta)\ \left(T'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - T'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\right) U'_\delta(\xi,\delta) +$$
$$+T(\xi,\delta)\ \left(M'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right) Y'_\delta(\xi,\delta))/\Psi_1, \quad (42)$$

$$\frac{\partial t}{\partial \xi} = \left(Y(\xi,\delta)\ M'_\xi(\xi,\delta)\ U^\alpha + T(\xi,\delta)\ U'_\xi(\xi,\delta)\right) \times$$
$$\times \left(Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right)/\Psi_1, \quad (43)$$

$$\frac{\partial t}{\partial \delta} = (Y(\xi,\delta)\ M'_\delta(\xi,\delta)\ U^\alpha + T(\xi,\delta)\ U'_\delta(\xi,\delta)) \times$$
$$\times \left(Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta) - Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta)\right)/\Psi_1,$$
$$\Psi_1 = (Y(Y\left(M'_\delta(\xi,\delta)\ T'_\xi(\xi,\delta) - M'_\xi(\xi,\delta)\ T'_\delta(\xi,\delta)\right) +$$
$$+T\left(Y'_\delta(\xi,\delta)\ M'_\xi(\xi,\delta) - Y'_\xi(\xi,\delta)\ M'_\delta(\xi,\delta)\right)) U^\alpha +$$
$$+T(Y\left(U'_\delta(\xi,\delta)\ T'_\xi(\xi,\delta) - U'_\xi(\xi,\delta)\ T'_\delta(\xi,\delta)\right) +$$
$$+T\left(Y'_\delta(\xi,\delta)\ U'_\xi(\xi,\delta) - Y'_\xi(\xi,\delta)\ U'_\delta(\xi,\delta)\right))). \quad (44)$$

Analogously to (17), the function $T(\xi,\delta)$ can be found from (7). Theorem 5 is analogous to Theorem 1 and can be proved in a complete analogy with it.

Theorem 6.

Let the system (5)–(7), (10), (40) be given. Then, after the change of variables, the equation for the function of the first derivative $Y(\xi,\delta)$ in new variables can be separated from the other equations and depends only on the function $U(\xi,\delta)$ and its derivatives:

$$\frac{\partial}{\partial\delta}\left(U(\xi,\delta)^\alpha\left(\frac{\partial}{\partial\delta}\left(YY'_\delta\right)/U'_\delta\right)/U'_\delta\right) = (Y'_\delta U'_\xi - Y'_\xi U'_\delta)/Y^2. \quad (45)$$

The change of variables for the function $Y(\xi,\delta)$ has the form

$$Y(\xi,\delta) = \sqrt{G(\eta,\xi)}\Big|_{\eta=U(\xi,\delta)},\ \eta = U(\xi,\delta),\quad t(\xi,\delta) = \xi. \quad (46)$$

Then there exists an exact solution of the Harry Dym equation (39) described by the formulas

$$M(\xi,\delta) = G'_\eta(\eta,\xi)/2\Big|_{\eta=U(\xi,\delta)},\quad T(\xi,\delta) = -\eta^\alpha\sqrt{G(\eta,\xi)}G''_{\eta\eta}(\eta,\xi)/2\Big|_{\eta=U(\xi,\delta)},$$
$$x(\xi,\delta) = X(\eta,\xi)\Big|_{\eta=U(\xi,\delta)},\quad \frac{\partial\ X(\eta,\xi)}{\partial\xi} = \eta^\alpha G''_{\eta\eta}/2,\quad \frac{\partial X(\eta,\xi)}{\partial\eta} = 1/\sqrt{G(\eta,\xi)},$$
$$\det J = -U'_\delta(\xi,\delta)/G(\eta,\xi)^{1/2}\Big|_{\eta=U(\xi,\delta)}, \quad (47)$$

after the change of variables (46) with a smooth function $Y(\xi,\delta)$ determined by the equations

$$G(\eta,\xi)^{-3/2}\frac{\partial G(\eta,\xi)}{\partial \xi}+\frac{\partial}{\partial \eta}\left(\eta^{\alpha}\frac{\partial^2 G(\eta,\xi)}{\partial \eta^2}\right)=0. \tag{48}$$

Here $U(\xi,\delta)$ is an arbitrary trice continuously differentiable function.

Proof. The proof of Theorem 6 is analogous to that of Theorem 3. Consider a fixed point $t(\xi,\delta)=\xi,\ t'_\delta=0$ in (44). Then from (40) we get relations

$$M'_\delta(\xi,\delta)=-U'_\delta(\xi,\delta)TU^{-\alpha}/Y,\ \ t(\xi,\delta)=\xi,\ M(\xi,\delta)=Y'_\delta Y(\xi,\delta)/U'_\delta(\xi,\delta). \tag{49}$$

Equating the derivatives $M'_\delta\equiv M'_\delta$ from (49), we get

$$T(\xi,\delta)=-U^{\alpha}Y\left(\frac{\partial}{\partial\delta}\left(YY'_\delta/U'_\delta\right)\right)/U'_\delta. \tag{50}$$

Using equation (43) $t'_\xi=1$, we obtain

$$\frac{\partial}{\partial\delta}\left(T(\xi,\delta)/Y(\xi,\delta)\right)+\left(Y'_\delta U'_\xi-Y'_\xi U'_\delta\right)/Y^2=0. \tag{51}$$

Eliminating the function T from (50) and (51), one obtains (45) and (51). The solvability conditions (25) again imply equations (45) and (48). Put $x(\xi,\delta)=X(U(\xi,\delta),\xi)$. Then we get relations (47) for derivatives. From equations (47) one can also obtain the solvability condition $\frac{\partial^2 X(\eta,\xi)}{\partial\eta\partial\xi}=\frac{\partial^2 X(\eta,\xi)}{\partial\xi\partial\eta}$ that must hold. This again imply equations (45), (48). *Theorem 6 is proved.*

Note that equation (45), (48) are new in the theory of the HD equation and that they trice appear in the proof of Theorem 6. This is not so in the case of the Harry Dym–Burgers equation with dissipation.

7. A second class of exact solutions to the Harry Dym equation.

This class is more complicated than a first one.

Theorem 7.

Let the system (5), (6), (40), (10) be given. Then an exact solution of the HD equation has the form

$$Y(\xi,\delta)=\sqrt{G(\eta,\delta)}\Big|_{\eta=U(\xi,\delta)},\ \eta=U(\xi,\delta),\ x(\xi,\delta)=\delta, \tag{52}$$

$$\begin{aligned}&M(\xi,\delta)\ \stackrel{\text{def}}{=}\ M_0(U(\xi,\delta),\delta)\Big|_{U(\xi,\delta)=\eta},\\ &M_0(\eta,\delta)=\left(G'_\delta(\eta,\delta)/\sqrt{G(\eta,\delta)}+G'_\eta(\eta,\delta)\right)/2,\\ &t(\xi,\delta)=\tau(U(\xi,\delta),\delta).\end{aligned} \tag{53}$$

The function $T(\xi,\delta)$ is determined from (7): $T(\xi,\delta) = T_0(U(\xi,\delta),\delta)\Big|_{U(\xi,\delta)=\eta}$. Then $T_0(\eta,\delta) =$
$= (1/G'_\delta(\eta,\delta))\Big(\eta^\alpha\sqrt{G(\eta,\delta)}(M'_{0\delta}(\eta,\delta)(-2M_0+G'_\eta(\eta,\delta))-G'_\delta(\eta,\delta)M'_{0\eta}(\eta,\delta))\Big) =$
$= \eta^n(G'_\delta)^2/(4\sqrt{G(\eta,\delta)}^3-\eta^\alpha G''_{\delta\delta}/(2\sqrt{G})+\eta^\alpha G'_\delta G'_\eta/(4\,G)-\eta^\alpha G''_{\delta\eta}-\eta^\alpha\sqrt{G}G''_{\eta\eta}/2.$
Then,in
new variables, the equation for the function of the first derivative $Y(\xi,\delta) =$
$= \sqrt{G(\eta,\delta)}\Big|_{U(\xi,\delta)=\eta}$ can be separated from the other equations and depends only on the function $U(\xi,\delta)$ and its derivatives:

$$\frac{\partial^3 G(\eta,\delta)}{\partial\eta^3} + 3\frac{\partial^3 G(\eta,\delta)}{\partial\eta\partial\delta^2}/G + 3\frac{\partial^3 G(\eta,\delta)}{\partial\eta 2\partial\delta}/\sqrt{G(\eta,\delta)} + \alpha G''_{\eta\eta}/\eta +$$
$$+\left(2\alpha/(\eta\sqrt{G}) - 3G'_\delta/(2G^2) - 3G'_\eta/(2\sqrt{G^3})\right)\frac{\partial^2 G(\eta,\delta)}{\partial\eta\partial\delta} +$$
$$+3\frac{\partial G(\eta,\delta)}{\partial\delta}\left(\frac{\partial G(\eta,\delta)}{\partial\eta}\right)^2/(4\sqrt{G^5}) + \frac{\partial^3 G(\eta,\delta)}{\partial\delta^3}/\sqrt{G^3} - \alpha(G'_\delta)^2/(2\;\eta\;G^2) +$$
$$+\left(-\alpha G'_\delta/(2\eta\sqrt{G^3}) + 3(G'_\delta)^2/(2G^3) - 3G''_{\delta\delta}/(2G^2)\right)\frac{\partial G(\eta,\delta)}{\partial\eta} +$$
$$+\left(\alpha/(\eta G) - 3G'_\delta/(2G^{5/2})\right)G''_{\delta\delta} + 3(G'_\delta)^3/(4\sqrt{G^7}) = 0. \tag{54}$$

The derivatives of the function $\tau(\eta,\delta)$ have the form

$$\tau'_\delta(\eta,\delta) = 4\eta^{-\alpha}G^{3/2}(\eta,\delta)/\Psi_3,\;\; \tau'_\eta(\eta,\delta) = -4\eta^{-\alpha}\sqrt{G^2(\eta,\delta)}/\Psi_3,$$
$$\Psi_3 = (G'_\delta)^2 + \sqrt{G}G'_\delta G'_\eta - 2G\left(G''_{\delta\delta} + 2\sqrt{G}G''_{\delta\eta} + GG''_{\eta\eta}\right). \tag{55}$$

Here $U(\xi,\delta)$ is an arbitrary trice continuously differentiable function.

Proof. The proof is analogous to that of Theorem 4.

Put $x(\xi,\delta) = \delta,\; x'_\xi(\xi,\delta) = 0,\; x'_\delta(\xi,\delta) = 1$. Then (41) and (42) imply two equations. The expression for the function $T(\xi,\delta)$ follows from (7) analogously to (37), (38). Using the changes of variables (52),(53) we get the relation $T_0(\eta,\delta)$ and then (53), as well as equation (54). We calculate the first and the second derivatives of these functions, which will be later used to simplify the expressions obtained. Using the changes of variables (52), (53), we get relations between the derivatives $\tau'_\delta(\eta,\delta),\; \tau'_\eta(\eta,\delta)$ in (55). $\tau'_\delta(\eta,\delta),\; \tau'_\eta(\eta,\delta)$. The solvability conditions (25) in variables $\eta,\;\delta$ also must hold: $\frac{\partial^2\tau(\eta,\delta)}{\partial\eta\partial\delta} = \frac{\partial^2\tau(\eta,\delta)}{\partial\delta\partial\eta}$. Thus, we once more obtain equation (54). *Theorem 7 is proved.*

Remark 3. Theorems 4, 5, 7, 8 deal only with one branch of the solution, with the exception of the example, where it was necessary to describe the both branches. The formulas for a negative branch of the solution $Y(\xi,\delta) = -\sqrt{G(\eta,\xi)}$ are deduced analogously.

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